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# A Macroscopic Device for Quantum Computation

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**Abstract** We show how a compound system of two entangled qubits in a non-product state can be described in a complete way by extracting entanglement into an internal constraint between the two qubits. By making use of a sphere model representation for the spin $1/2$, we derive a geometric model for entanglement. We illustrate our approach on 2-qubit algorithms proposed by Deutsch, respectively Arvind. One of the advantages of the 2-qubit case is that it allows for a nice geometrical representation of entanglement, which contributes to a more intuitive grasp of what is going on in a 2-qubit quantum computation.

**Keywords** Quantum computation · Entanglement · Macroscopic quantum models

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## 1 Introduction

The theory of quantum computation has gained much interest over the past two decades, both for the possible useful applications of quantum systems in computation and cryptography, as well as for the insights it provides in the foundations of quantum mechanics. Deutsch defined the Quantum Turing Machine and proposed the first quantum algorithm for a 2-qubit quantum computer, which could solve a problem faster than possible on a classical digital computer [1]. Shor presented a factoring algorithm which allows to break cryptographic codes faster than by any known means classical possible [2]. Grover showed the existence of a fast quantum algorithm for database search [3]. However, the actual physical implementation of quantum computers still encounters many technical problems, and up to now only quantum computers with a limited number of qubits have been built. In this paper we explore an alternative to this search for a microscopic quantum computer, namely to consider macroscopic models simulating a quantum computation.

We adopt an operational approach to quantum mechanics in which a physical entity is described by its set of states, its set of properties and a relation of 'actuality' between these two sets which expresses which properties are actual when the system is in a specific state [4]. Following this operational approach, it is a natural step to consider models which exhibit 'quantum behavior' on a structural level, even if they are not microscopic in size. This has lead to a 'quantum-like' sphere model which has a structure which is isomorphic to the spin structure for a spin $1/2$ [5] and another system which entails a structure isomorphic to the structure of two spin $1/2$ in the entangled singlet state [6]. This model has been elaborated by showing that an arbitrary tensor product state representing two entangled qubits can be described in a complete way by a specific internal constraint between the ray or density states of the two qubits, which describes the behavior of the state of one of the spins if measurements are executed on the other spin [7]. This means that in principle one can represent any entangled state of a 2-qubit quantum computer with this model. We illustrate this on two quantum algorithms, the first is the algorithm of Deutsch [1] and the second is the algorithm proposed by Arvind et al. [8, 9]. Since any $n$-qubit interaction can be decomposed into 2-qubit gates and unary operations [10] we argue that our representation of 2-qubit entanglement contributes to a better understanding of the $n$-qubit quantum computer. This is very important since at this stage it is not at all clear how entanglement can be exploited in a systematic way. In this sense, our geometric model is a step towards the better understanding of the exploitation of 2-qubit entanglement in a quantum algorithm.

## 2 An Operational Approach to Quantum Mechanics

The basic notions for the description of a physical entity $S$ are as follows [4]. First, we consider that at any moment the entity $S$ is in a (known or unknown to the observer) state $p \in \Sigma$. Also, $S$ has a set of properties $\mathcal{L}$, defined by the set of available experiments which can be performed on $S$. A property $a$ is either 'actual' or 'potential' for the entity $S$, meaning that if the property $a$ is actual in the state $p$, then whenever one would perform the corresponding experiment, one finds the positive outcome with certainty. Between the set of states and (power)set of properties is a relation $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ of actuality that maps each state $p \in \Sigma$ onto the set $\xi(p)$ of those properties that are actual in this state. Depending on the nature of the entity $S$, one obtains a different structure on the set of states $\Sigma$, the set of properties $\mathcal{L}$ and the relation between these two sets. Hence, if we are only concerned with the structural behaviour of an entity, we can focus on the triple $(\Sigma, \mathcal{L}, \xi)$, called a State Property

Space (**SPS**). If one considers the **SPS** of a quantum entity, one observes that the mathematical structure of the **SPS** obeys certain 'quantum axioms'. Conversely, one could start from an abstract **SPS**, and by imposing a suitable set of axioms one can derive a quantum structure on the set of properties, i.e. it becomes isomorphic with the Hilbert space representation of quantum mechanics. In this operational approach, a physical system is determined by the structure on its set of states and properties, and the actuality map $\xi$ between these two. This means that it is not necessary (or even meaningful) to make a distinction between quantum and classical systems based on their size (classical macroscopic versus microscopic quantum) but rather by the different structure of their **SPS**. Within this view various macroscopic models have been constructed which have a quantum structure on their set of states and properties, and a quantum probability distribution on the set of outcomes. It is one of these models which will be used to model the 2-qubit entangled states.

## 3 Sphere Model for the Single Qubit

The sphere model is a generalization of the Bloch sphere representation such that also the measurements are represented [5, 11]. In the Bloch representation, a qubit $|\psi\rangle = \cos\frac{\theta}{2}e^{\frac{-i\phi}{2}}|0\rangle + \sin\frac{\theta}{2}e^{\frac{i\phi}{2}}|1\rangle$ is represented by the point $u(1,\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on the surface of a 3-dimensional sphere. The main differences with the standard use of the Bloch representation are as follows. First, in our approach all points of the Bloch sphere represent states of the spin, such that points on the surface correspond to pure states, while interior points correspond to density states. This is because an arbitrary point $u(r,\theta,\phi)$, $r \in [0,1], \theta \in [0,\pi], \phi \in [0,2\pi)$, of the Bloch sphere can be expressed as a convex linear combination

$$u(r,\theta,\phi) = ru(1,\theta,\phi) + (1-r)u(0,\theta,\phi)$$

from which follows the corresponding density state

$$\begin{aligned} D(r,\theta,\phi) &= rD(1,\theta,\phi) + (1-r)D(0,\theta,\phi) \\ &= \frac{1}{2}\begin{pmatrix} 1+r\cos\theta & r\sin\theta e^{-i\phi} \\ r\sin\theta e^{i\phi} & 1-r\cos\theta \end{pmatrix}. \end{aligned}$$

In this expression $D(1,\theta,\phi) = |\psi\rangle\langle\psi|$ is the usual density state representation of a pure state, while $D(0,\theta,\phi)$ is the density matrix representing the center of the sphere. The spin up state (the pure state $|0\rangle$) corresponds with the 'North pole' $D(1,0,\phi) = |0\rangle\langle 0|$, while the spin down state (hence pure state $|1\rangle$) is represented by the 'South pole' $D(1,\pi,\phi) = |1\rangle\langle 1|$.

Next to this, the sphere model allows a representation of measurements (see Fig. 1). The experiments $e_u$ are defined as follows. We put an elastic of length 2 centered in the origin $o$ of the sphere $S^2$ between the point $u$ and its antipode $-u$. Let us denote the segment between $u$ and $-u$ with the interval $[-u,u]$. Next, the particle falls from its initial position $p$ orthogonally onto the interval $[-u,u]$ in the point $p'$ where it stays attached to the elastic. Then the elastic breaks randomly in some point $\lambda \in (-u,u)$ such that two possibilities can occur. If the elastic breaks between $p'$ and $-u$, the elastic will pull the point particle towards $u$ where it stays attached and the experiment is said to yield the outcome 'spin up'. If on the other hand the elastic breaks between $u$ and $p'$, then the elastic will pull the particle towards $-u$, where it stays attached, and the measurement is said to yield outcome 'spin down'. To make the description of the experiment $e_u$ complete, one could specify that in

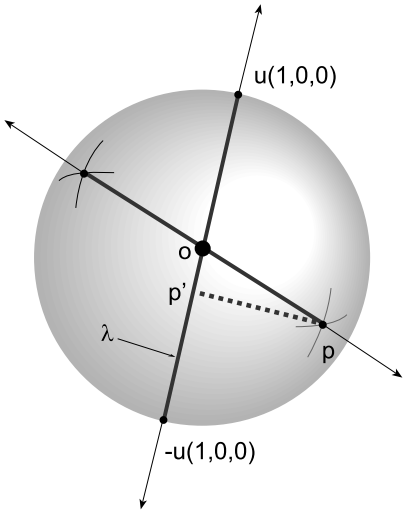


**Fig. 1** The macroscopic spin 1/2 model

the event that the elastic breaks at exactly the point $p'$ where the particle is attached, we assume that the measurement always yields the outcome 'spin up'. Notice however that this event has measure zero to occur, and in such sense it is physical irrelevant with respect to the resulting probability distribution over the set of outcomes. Let $\theta$ denote the angle between the state $p$ of the system and the direction $u$ of the measurement device. The probability for outcome 'spin up' is given by the length of the elastic between the projection point $p'$ and the point $-u$, normalized by dividing by the total length of the elastic. This yields following probability $P(u \mid p)$ for the 'spin up' outcome and corresponding state transition from initial state $p$ towards final state $u$, eigenstate of the 'spin up' outcome:

$$P(u \mid p) = \frac{\cos\theta + 1}{2} = \cos^2 \frac{\theta}{2}.$$

Similarly we can calculate the probability for the outcome 'spin down' as

$$P(-u \mid p) = \frac{1 - \cos\theta}{2} = \sin^2 \frac{\theta}{2}.$$

These probabilities coincide with the quantum probabilities for a spin experiment on a spin-$\frac{1}{2}$ particle. Note that if one would have knowledge about where the elastic breaks, the measurement procedure happens deterministic. If we call $e_u^\lambda$ the measurement that consists in performing $e_u$ and such that the elastic breaks in the point $\lambda$ for some $\lambda \in (-u, u)$, then, each time $e_u$ is performed, it is actually one of the $e_u^\lambda$ that takes place. We do not control this, in the sense that the $e_u^\lambda$ are really 'hidden measurements' that we cannot choose to perform. The probability $\mu(e_u, p, o_1)$ that the experiment $e_u$ gives the outcome $o_1$ if the entity is in state $p$ is a randomization over the different situations where the hidden measurements $e_u^\lambda$ gives the outcome $o_1$ with the entity in state $p$. More generally, one could regard any quantum experiment as a class of 'hidden measurements' such that each hidden measurement by itself is deterministic, but a lack of knowledge about which hidden measurement is actually going on leads to a lack of knowledge on the level of the measurement outcomes, i.e. quantum probability can be explained as due to an uncontrollable (and irreducible) lack of knowledge on the interaction between the measurement device and the

system. This interpretation of quantum mechanics is called the 'hidden measurement interpretation' of quantum mechanics. Since it locates the lack of knowledge in the measurement interaction, it is contextual from the very start and goes beyond the reach of standard 'no go theorems' for hidden variable theories for quantum mechanics [12, 13].

## 4 Macroscopic Model of Two Entangled Spin 1/2

Putting together two sphere models one can construct a macroscopic model for a compound system of two entangled spin $1/2$ in the singlet state [6]. The compound system consists of two point particles in the center of two sphere models for a single spin $1/2$ which are connected by a rigid but extendable rod, centered in $c$, i.e. in the middle of the line connecting the two point particles (see Fig. 2a). The joint experiment $e(a, b)$ is performed as follows. First, the two spheres reach the measurement apparatuses, with measurement direction $a$ respectively $b$ chosen by the experimenter. The spin measurements $e_a$ and $e_b$ are defined similarly as for the single sphere model. When one side is measured, the measurement apparatus draws one of the entities to one of the two possible outcomes with probability $1/2$ (because initially the point particle is in the center of the sphere) (see Fig. 2b). Because of the rod the other entity is drawn toward the opposite side of the sphere as compared with the first entity (see Fig. 2c, in which we consider for the sake of illustration the event that $e_a$ has yielded outcome 'spin up'). The connecting rod is then taken away and the second spin measurement is performed (see Fig. 2d). The measurement $e(a, b)$ has 4 possible outcomes $(a, b)$, $(a, -b)$, $(-a, b)$ and $(-a, -b)$. One finds that the expectation value $\mathbb{E}(a, b) = -a.b$, i.e. this corresponds with the expectation value for the spin measurements along directions $a, b$ on a compound system of two entangled spin $1/2$ particles in the singlet state [14].

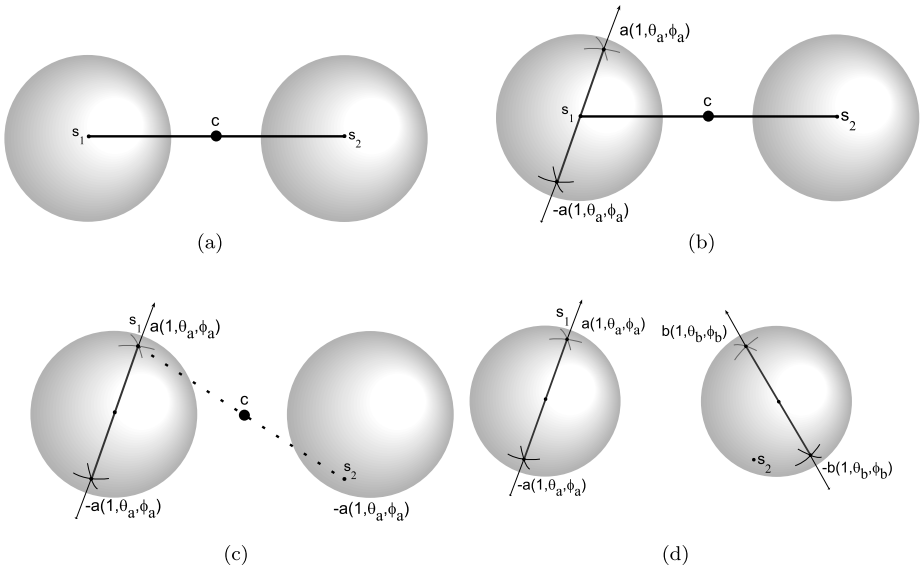


**Fig. 2** (**a**) The initial configuration of the macroscopic model of two entangled spin $1/2$ in the singlet state. (**b**) The measurement $e_a$ is performed on the left point particle, drawing it towards the point $a$. (**c**) By means of the rigid rod, the right point particle is drawn towards the point $-a$ on its sphere. Next, the connecting rod is taken away. (**d**) Finally, the measurement $e_b$ is performed on the right point particle

It should be mentioned that other proposals of macroscopic simulations of quantum-like behaviour have been recently published in [15, 16] (see also references to previous publications by the same authors cited therein).

It is not obvious that the 'faster than light aspect' identified in the quantum experiments is also present in the connected rod singlet model. A classical model, for example using a rigid rod, will only describe physical interactions of causal nature, propagating with a speed smaller than the speed of light, and hence a priori will not be able to model the speed effects within the quantum experiment. However it is not the speed effects present in this entanglement correlation which is vital in a quantum computation, but rather its algebraic structure. In this sense, if one could build a model of 2-qubit entanglement (even with 'slower than light' implementation), it is this possibility to implement entanglement in the compound system which makes it non-classical, and therefore impossible to implement by classical means.

## 5 Geometrical Representation of two Entangled Spin 1/2

### 5.1 Constraint Functions and the Schmidt Diagonal Form

In the previous section we discussed a compound system consisting of two connected sphere models which represents the singlet state of two entangled spin 1/2. In general, in a 2-qubit quantum algorithm the register will be in other entangled states as well. In [7] we show how this problem is solved for the coupled sphere models by introducing constraint functions. Let us briefly recall how these functions are introduced and some of their properties.

A system of two entangled spin $\frac{1}{2}$ is described by means of a unit vector $|\psi\rangle \in \mathbb{C}_1^2 \otimes \mathbb{C}_2^2$, which can always be written as $|\psi\rangle = \sum_{ij} \lambda_{ij} |e_1^i\rangle \otimes |e_2^j\rangle$, where $\lambda_{ij} \in \mathbb{C}$, and $\{|e_1^i\rangle\}$ and $\{|e_2^j\rangle\}$ are bases of $\mathbb{C}_1^2$ and $\mathbb{C}_2^2$ respectively. Following von Neumann [12], when a measurement is performed on the first spin, it collapses into a spin state described by the unit vector $|x_1\rangle \in \mathbb{C}_1^2$ corresponding with the observed outcome, thus transforming the entangled state $|\psi\rangle$ into $(P_{|x_1\rangle} \otimes I)(|\psi\rangle)$, where $P_{|x_1\rangle}$ is the orthogonal projector on $|x_1\rangle$ in $\mathbb{C}_1^2$, and $I$ is the unit operator in $\mathbb{C}_2^2$. The result is that the entangled spins end up in the product state $|x_1\rangle \otimes \sum_{ij} \lambda_{ij} \langle x_1, e_1^i\rangle |e_2^j\rangle$. This means that as a consequence of the measurement on the first spin, collapsing its state to $|x_1\rangle$, the second spin collapses to the state $\sum_{ij} \lambda_{ij} \langle x_1, e_1^i\rangle |e_2^j\rangle$. In an analogous way we can show that if a measurement is performed on the second spin, resulting in a collapse to the state $x_2 \in \mathbb{C}_2^2$, the state of the first spin becomes $\sum_{ij} \lambda_{ij} \langle x_2, e_2^j\rangle |e_1^i\rangle$. Because of this, we arrive at the following definition.

**Definition 1** (Constraint Functions) The constraint functions $F_{12}(\psi)$ and $F_{21}(\psi)$ related to $\psi$ are defined in the following way:

$$F_{12}(\psi): \quad \mathbb{C}_1^2 \to \mathbb{C}_2^2 : |x_1\rangle \mapsto \sum_{ij} \lambda_{ij} \langle x_1, e_1^i\rangle |e_2^j\rangle,$$

$$F_{21}(\psi): \quad \mathbb{C}_2^2 \to \mathbb{C}_1^2 : |x_2\rangle \mapsto \sum_{ij} \lambda_{ij} \langle x_2, e_2^j\rangle |e_1^i\rangle.$$

The constraint functions map the state where one of the spins collapses to by a measurement to the state that the other spin collapses to under influence of the entanglement correlation. A detailed study of the constraint functions gives us a complete picture of how

the entanglement correlation works as an internal constraint. It can be shown that the constraint functions are canonically defined, i.e. the definition of $F_{12}(\psi)$ and $F_{21}(\psi)$ do not depend on the chosen bases. Also, $F_{21}(\psi) \circ F_{12}(\psi) = D_1(\psi) \equiv tr_{\mathbb{C}_1^2}|\psi\rangle\langle\psi|$ i.e., the partial trace density matrix over $\mathbb{C}_2^2$. Similarly, $F_{12}(\psi) \circ F_{21}(\psi) = D_2(\psi)$ is the partial trace density matrix over $\mathbb{C}_1^2$.

To derive a complete view of how entanglement works as an internal constraint for a 2-particle system, we recall the relation between the Schmidt diagonal form [17] and the constraint functions. Let us choose the following base in $\mathbb{C}_1^2$:

$$\{|x_1^1\rangle, |x_1^2\rangle\} = \left\{\left(\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \sin\frac{\theta}{2}e^{\frac{i\phi}{2}}\right), \left(-i\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}, i\cos\frac{\theta}{2}e^{\frac{i\phi}{2}}\right)\right\}.$$

With respect to this basis, the expression for a general density matrix becomes

$$D_1(\psi) = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}.$$

Define now the following basis $\{|x_2^1\rangle, |x_2^2\rangle\}$ in $\mathbb{C}_2^2$

$$\{|x_2^1\rangle, |x_2^2\rangle\} = \left\{\frac{\sqrt{2}}{\sqrt{1+r}}F_{12}(\psi)(|x_1^1\rangle), \frac{\sqrt{2}}{\sqrt{1-r}}F_{12}(\psi)(|x_1^2\rangle)\right\}.$$

One finds that $|x_2^1\rangle$ and $|x_2^2\rangle$ are normalized eigenvectors of $D_2(\psi)$ with eigenvalues $\frac{1+r}{2}$ and $\frac{1-r}{2}$ respectively. Therefore, with respect to the basis $\{|x_2^1\rangle, |x_2^2\rangle\}$, $D_2(\psi)$ is expressed as

$$D_2(\psi) = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}.$$

Finally, one obtains [7] that the Schmidt diagonal form of $|\psi\rangle$ is given by

$$|\psi\rangle = \frac{\sqrt{1+r}}{\sqrt{2}}|x_1^1\rangle \otimes |x_2^1\rangle + \frac{\sqrt{1-r}}{\sqrt{2}}|x_1^2\rangle \otimes |x_2^2\rangle.$$

### 5.2 Geometrical Representation of Entanglement

Let us remark that in general $F_{12}(\psi)$ does not conserve orthogonality, except for the basis vectors $\{|x_1^1\rangle, |x_1^2\rangle\}$ and in the case that $\psi$ is a product state. Also, $F_{12}(\psi)$ does not conserve the norm, since $\|F_{12}(\psi)(|x\rangle)\|^2 = \frac{1}{2}(1 + r\cos\theta)$. This means that we have to look at the *normalized* image of a vector $|x\rangle = x(\theta_1, \phi_1)$. This vector is mapped to $|y\rangle = y(\theta_2, \phi_2) = \frac{1}{\|F_{12}(\psi)(|x\rangle)\|}F_{12}(\psi)(|x\rangle)$. One derives the following expression:

$$y(\theta_2, \phi_2) \cdot x_2^1(\theta, \phi) = \frac{r + \cos\theta_1}{1 + r\cos\theta_1}$$

from which one can deduce that straight lines containing the center $o_1$ on the first sphere are mapped to straight lines through the point $u(r, 0, 0)$ in the second sphere. This gives a geometrical representation of 'stretching' on the second sphere (see Fig. 3). Hence the entanglement correlation is represented by a rotation (reflecting the basis vectors in the Schmidt diagonal form) and a stretching depending on the parameter $r$, i.e. 'amount of entanglement'.

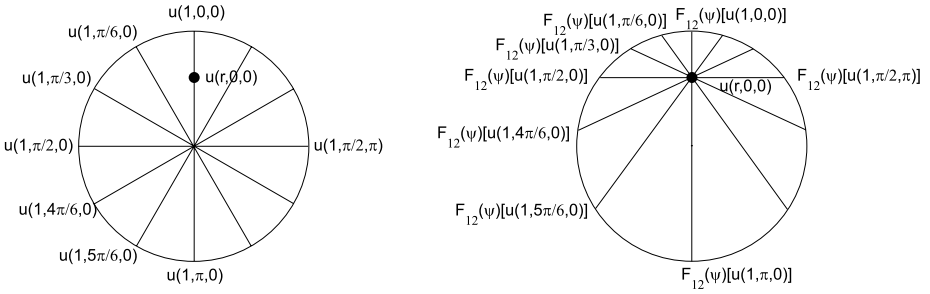


**Fig. 3** Geometrical representation of two entangled spin 1/2

Vice versa, given a general entangled state $|\psi\rangle$ expressed in the canonical basis, we can derive the Schmidt diagonal form using the constraint functions. From the reduced density matrix $D_1(\psi) = F_{21}(\psi) \circ F_{12}(\psi)$ one can calculate the basis vectors $\{|x_1^1\rangle, |x_1^2\rangle\}$, and consequently also $\{|x_2^1\rangle, |x_2^2\rangle\}$. Also, expressing the reduced density matrix $D_1(\psi)$ in its eigenvector basis $\{|x_1^1\rangle, |x_1^2\rangle\}$ we obtain the value for the parameter $r$, i.e. the amount of entanglement.

## 6 Geometrical Representation of 2-Qubit Algorithms

### 6.1 Deutsch Problem

Deutsch problem is as follows [1]: given a $\{0, 1\}$-valued function $f$ defined on a two-point domain $\{0, 1\}$, determine with a single call of the 'oracle' whether $f$ is constant or balanced. This problem cannot be solved on a classical digital computer. However, on a quantum computer this is possible. Consider preparing a quantum register of two qubits in an input state:

$$\psi_0 = (H \otimes H)|01\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|00\rangle + |10\rangle - |01\rangle - |11\rangle}{2}. \tag{1}$$

Calling the oracle corresponds with applying a suitable unitary transformation $U_f$ which maps $|xy\rangle$ onto $U_f|xy\rangle = |x(y \oplus f(x))\rangle$, where $x, y \in \{0, 1\}$ and $\oplus$ is the sum modulo 2. Denoting $\overline{f(i)} = f(i) \oplus 1$ and noticing that $|f(i)\rangle - |\overline{f(i)}\rangle = (-1)^{f(i)}(|0\rangle - |1\rangle)$ one obtains that $\psi_1 = U_f(\psi_0) = \frac{|0f(0)\rangle + |1f(1)\rangle - |0\overline{f(0)}\rangle - |1\overline{f(1)}\rangle}{2}$ which can be written as a product state:

$$\psi_1 = \left(\frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}}\right) \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{2}$$

After applying a Hadamard transformation $H \otimes H$ on both qubits we obtain the following state:

$$\psi_2 = (H \otimes H)U_f(\psi_0) = (-1)^{f(0)}[(1 - (f(0) \oplus f(1)))|0\rangle + (f(0) \oplus f(1))|1\rangle] \otimes |1\rangle. \tag{3}$$

- If $f$ is constant, $f(0) = f(1)$ and the state $\psi_2$ can be rewritten as:

$$\psi_{2,c} = (-1)^{f(0)}|0\rangle \otimes |1\rangle \tag{4}$$

i.e. if $f$ is constant, then the value of the first qubit is $|0\rangle$.

- If $f$ is balanced, $f(1) = f(0) \oplus 1$ then the state $\psi_2$ can be rewritten as:

$$\psi_{2,b} = (-1)^{f(0)}|1\rangle \otimes |1\rangle \tag{5}$$

such that if $f$ is balanced, then the first qubit is $|1\rangle$.

This shows that a quantum computer solves Deutsch problem with 1 call of the oracle. However, it does not achieve this task by calculating both values $f(0)$ and $f(1)$ 'parallel' and compare them simultaneously, but rather by acting on the state of the 2-qubit system as a whole.

### 6.2 Deutsch Problem on the Geometric Model

The intermediate states in Deutsch algorithm are given by the product states (1–5) such that one can represent these states in a trivial way on the geometric model (no entanglement is present, so the state of the compound system can be represented by the states of the individual qubits by their Cartesian product). Therefore, in the next section we consider a 2-qubit algorithm exploiting non-product states, such that the geometrical representation by two correlated sphere models becomes less trivial.

### 6.3 Arvind Problem

Arvind's problem is the following [8, 9]: given a $\{0, 1\}$-valued function $f$ defined from a 2-qubit domain space to a one-bit range space (i.e. $f(x) : \{0, 1\}^2 \to \{0, 1\}$), determine with 2 calls of the 'oracle' whether $f$ is even or odd. Since there are 4 possible input values $(00), (01), (10), (11)$ each of which has 2 possible output values $(0, 1)$ there are 16 possible functions $f$. These can be divided into sub-classes depending on the number of values 0 respectively 1, i.e. $[0, 4], [1, 3], [2, 2], [3, 1], [4, 0]$. One has 8 'even' functions ($[0, 4], [2, 2], [4, 0]$) and 8 'odd' functions ($[1, 3], [3, 1]$). On a classical computer Arvind's problem is impossible to solve, since it would require calling the oracle four times. However, Arvind et al. have shown how to solve this problem on a quantum computer. Again, calling the oracle corresponds with applying a unitary transformation $U_f : |x\rangle_{\text{2-qubit}} \to (-1)^{f(x)}|x\rangle_{\text{2-qubit}}$

$$U_f = \begin{pmatrix} (-1)^{f(00)} & 0 & 0 & 0 \\ 0 & (-1)^{f(01)} & 0 & 0 \\ 0 & 0 & (-1)^{f(10)} & 0 \\ 0 & 0 & 0 & (-1)^{f(11)} \end{pmatrix}.$$

There are 16 possible $U_f$ matrices, of which 8 are separable and 8 are non-separable, i.e. they are entangling, e.g. the matrix with diagonal entries $[1, 1, 1, -1]$. One could notice that the subclass of even functions $[0, 4], [4, 0], [2, 2]$ reflects the problem of constant or balanced functions in Deutsch problem. Hence it should come as no surprise that for even $f$ the matrix $U_f$ is separable. However, for odd $f$ the matrix $U_f$ is non-separable such that entanglement is necessary to solve Arvind's problem.

Let the quantum register of two qubits be prepared in input state $\psi_0 = |00\rangle$, then the consecutive unitary transformations in Arvind's algorithm are as follows:

$$\psi_0 = |00\rangle \to (H \otimes H)U_f(1 \otimes H)U_f(H \otimes H)|00\rangle$$

from which the consecutive 'intermediate' states in Arvind's algorithm can be derived. The final state (prior to measurement) is given by:

$$\psi_5 = \frac{1}{2\sqrt{2}}\big[\big((-1)^{f(00)+f(01)} + (-1)^{f(10)+f(11)}\big)|00\rangle + 2|01\rangle + \big((-1)^{f(00)+f(01)} - (-1)^{f(10)+f(11)}\big)|10\rangle\big].$$

- If $f$ is even, $(-1)^{f(00)+f(01)} = (-1)^{f(10)+f(11)}$ such that the final state becomes

$$\psi_{5,e} = \frac{1}{\sqrt{2}}(\pm|00\rangle + |01\rangle).$$

- If $f$ is odd, $(-1)^{f(00)+f(01)} = -(-1)^{f(10)+f(11)}$ and the final state becomes

$$\psi_{5,o} = \frac{1}{\sqrt{2}}(\pm|10\rangle + |01\rangle).$$

It is important to notice that although $\psi_{5,e} \neq \psi_{5,o}$, these states are *not* orthogonal, which makes it not possible to distinguish them in the standard approach of quantum computation. However, on a NMR based quantum computer these two states are distinguishable [9].

### 6.4 Arvind Problem on the Geometric Model

The density matrices for the intermediate states $\psi_i$ in Arvind's algorithm are given by:

$$D_1(\psi_i) = \begin{pmatrix} a_i & \overline{b_i} \\ b_i & 1-a_i \end{pmatrix}, \quad \begin{cases} a_i = \frac{1}{2}, i = 1,\ldots,4; a_5 = \frac{1}{4}[3 + (-1)^{\sum f(ij)}]; \\ b_1 = \frac{1}{2}; b_2 = b_3 = \frac{1}{4}[(-1)^{f(00)+f(10)} + (-1)^{f(01)+f(11)}]; \\ b_4 = \frac{1}{4}[1 + (-1)^{\sum f(ij)}]; b_5 = 0. \end{cases}$$

- If $f$ is even: $(-1)^{f(00)+f(10)} = (-1)^{f(01)+f(11)}$ the reduced density matrices $D_1(\psi_i)$ are given by:

$$D_1(\psi_1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad D_1(\psi_2) = D_1(\psi_3) = \begin{pmatrix} \frac{1}{2} & (-1)^{f(00)+f(10)}\frac{1}{2} \\ (-1)^{f(00)+f(10)}\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$
$$D_1(\psi_4) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad D_1(\psi_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

  such that the set of eigenstates for each matrix $D_1(\psi_i)$ is given by $\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\}, i = 1,\ldots,4$; the set of eigenstates for matrix $D_1(\psi_5)$ is $\{|0\rangle, |1\rangle\}$. Hence we obtain that all the consecutive states in Arvind's theorem are given by product states, making the representation by two coupled sphere models trivial since 'there is no entanglement correlation' present in these states.
- If $f$ is odd: $(-1)^{f(00)+f(10)} = -(-1)^{f(01)+f(11)}$ such that $b_i = 0, i = 2,\ldots,5$. The reduced density matrices $D_1(\psi_i)$ are hence given by:

$$D_1(\psi_1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad D_1(\psi_i) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad i = 2,\ldots,5.$$

Again, $\psi_1$ is a product state since applying the Hadamard gate on both qubits in the ground state $|00\rangle$ results in a product state. For $i = 2, \ldots, 5$ the reduced density matrices $D_1(\psi_i)$ correspond with an entangled state, namely the singlet state. However, there is no unique Schmidt diagonal form for the singlet state and hence no unique representation of the intermediate states in Arvind's algorithm on the geometric model for odd $f$. Nevertheless, the fact that these are singlet states also has an unexpected advantage, because these singlet states can be represented by the rigid rod model in a mechanistic way, making it possible to represent Arvind's algorithm 'directly' on the macroscopic singlet model.

## 7 Conclusions

We have shown how a compound system of two entangled qubits in a non-product state can be described in a complete way by extracting entanglement into an internal constraint between the two qubits. By making use of the sphere model representation for the spin 1/2, this allows for an easy to grasp visual support for the developed formalism. The resulting geometric model for entanglement allows for a rigorous method to follow its form and effect in quantum computations. This is very important since at this stage it is not at all clear how entanglement can be exploited in a systematic way. In this sense, our geometric model is a step towards the better understanding of the exploitation of 2-qubit entanglement in a quantum algorithm. We have discussed the 2-qubit quantum algorithms of Deutsch and Arvind and illustrated them on the geometric model. In future work the 2-qubit models should be generalized to the $n$-qubit quantum computer. Since any quantum system with a measurable set of outcomes allows a Hidden Measurement representation [5, 18], this approach should in principle be feasible. Also, it is still an open question whether one can find a physical (rod-like?) implementation of the constraint functions for the non-singlet states of the 2-qubit model and possible $n$-qubit generalizations thereof.

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